Developing Strategic Competence by Teaching Using the Common Core Mathematical Practices

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Imagine one of your mathematically proficient students. According to the National Research Council (NRC; 2001), that student would be proficient in five interwoven strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and a productive disposition toward mathematics. In fact, the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices [NGA Center] and Council of Chief State School Officers [CCSSO] 2010) used these five strands along with the National Council of Teachers of Mathematics (NCTM) Process Standards (2000) to develop the eight Standards for Mathematical Practice. As educators, we need to consider the mathematical practices that promote teaching for mathematical proficiency.

Now imagine a mathematically proficient teacher. This teacher would be proficient in these strands and could elicit these important strands of knowledge in students. But what does that look like in teachers? More specifically, what does strategic competence look like in skillful teachers? Our chapter describes a study that included a lesson study (Lewis 2002) conducted with classroom teachers and mathematics specialists and focused on developing teachers' strategic competence and strategies for modeling mathematical ideas.

Defining Strategic Competence and Teaching through Mathematical Practices

Strategic competence is one of the strands of mathematical proficiency. The National Research Council defines it as the "ability to formulate, represent, and solve mathematical problems" (2001, p. 116). This strand includes problem solving and problem formulation, which require solving a problem by representing it mathematically: numerically, mentally, symbolically, verbally, or graphically. The key attribute for people who have achieved strategic competence is flexibility in their problem-solving processes and strategies.

Strategic competence for teachers encompasses many important practice-based skills. For our study, strategic competence for teachers includes the ability to (a) formulate, represent, and solve problems; (b) model mathematical ideas; and (c) demonstrate representational fluency, that is, the ability to translate and connect within and among multiple representations with accuracy, efficiency, and flexibility (Lesh et. al 2003; Suh et al. 2012). In addition, a teacher who has strategic competence can intentionally use representations such as student-created diagrams, graphs, manipulative models, and numeric or verbal statements as pedagogical content tools (Rasmussen and Marrongelle 2006) to connect students' thinking while moving the mathematical agenda forward. The Association of Mathematics Teacher Educator's (AMTE) standards for pedagogical knowledge include the ability to "construct and evaluate multiple representations of mathematical ideas or processes; establish correspondences between representations; understand the purpose and value of doing so; and use various instructional tools, models, technology, in ways that are mathematically and pedagogically grounded" (2010, p. 4). Proficient teaching of mathematics, in terms of strategic competence, also requires teachers to be able to "plan effective instruction and solve problems that arise during instruction" (NRC 2001, p. 380).

We wanted to learn how a focus on strategic competence would translate into classroom practices and assessment strategies. We designed a study to explore the following research questions**:** (1) How do teachers use representations and models to elicit strategic competence in students? (2) How do mathematical knowledge and instructional practices come into play as teachers focus on developing students' strategic competence?

The study consisted of professional development and collaboratively planned lessons. The professional development took place at the end of the summer and into the fall semester, starting with a summer content institute that met for thirty hours and including five follow-up meetings (a total of fifteen hours) for lesson study. This led up to a full day of release time to conduct the research lessons where teachers gathered to observe the collaboratively planned lesson.

Teachers participated in lesson-study teams made up of four to six teachers led by a lesson-study facilitator. This chapter reports on one team of six teachers from grades 4 through 8 that planned the research lesson around a rational number problem called the Mango problem. After observing and debriefing the lesson, the other observers modified the lesson based on the discussion and taught a second cycle in their respective classrooms. This modified version of lesson study (Lewis 2002) allowed for these teachers to immediately refine the lesson for their grade level and share the data from their own classroom. The overarching research goal for the research lesson was enhancing teachers' strategic competence by helping them learn how to promote effective instructional practices and to elicit strategic competence in their students. The lessons focused on unitizing fractions and working backward to solve a problem.

We designed our professional development and the research lessons based on the current emphasis on the mathematical practices from the Common Core State Standards (NGA Center and CCSSO 2010) and the great need to enhance teachers' mathematics content and pedagogical knowledge. As we worked toward our goal of developing teachers with strategic competence, we saw many connections between the Common Core mathematical practices and the questioning prompts we encouraged in our lesson design, listed in fig. 8.1. For example, in our collaborative planning meeting, we spent time generating key questioning prompts that would elicit these mathematical practices. The high-yield instructional practices that we focused on were choosing rich tasks and maintaining that richness (Stein and Smith 2011) through questioning and engagement. In doing so, the teachers and the research team preplanned some prompts and questions that would elicit students' mathematics thinking (see fig. 8.1). In turn, this attention on questioning provided students opportunities to display their strategic competence, which allowed us opportunities to collect rich assessment data about student learning.

Mathematical Practices	Questioning prompts
(MP.1) Make sense of problems and persevere in solving them.	Does the problem make sense? Are students engaged in the act of "doing mathematics"? What do you need to find out? What information do you have? What strategies are you going to use? What can you do when you are stuck?
(MP.2) Reason abstractly and quantitatively.	What do the numbers in the problem mean? What are the relationships among the numbers in the problem? What is the relationship between the given problem and the mathematical representation? Can you generalize your pattern or strategy?
(MP.3) Construct viable arguments and critique the reasoning of others.	Do you agree? Why or why not? Does anyone have the same answer but a different way to explain it? How are some of my classmates strategies related and are some strategies more efficient than others? How does your partner's solution differ from yours?
(MP.4) Model with mathematics.	How is this math used in real world context? Can I model the problem using a picture or diagram? How could you think about this with numbers, words, pictures, and graphs? What do the variables in your model mean physically?
(MP.5) Use appropriate tools strategically.	What tools or technology can you use to solve the problem? Are certain manipulatives or representations more precise, efficient, or clearer than others?
(MP.6). Attend to precision.	What math vocabulary or concept and representations add accuracy and precision to student thinking? Can you justify why your method is precise?
(MP.7) Look for and make use of structure.	Are there steps that need to be taken to solve the problem? Is this problem related to a class of problems? Can you use a particular algorithmic process to solve this problem? What do you think is the rule? Are there other ways to represent the rule?
(MP.8) Look for and express regularity in repeated reasoning.	Do you see a pattern? Can you explain the pattern? Is there a pattern that can be generalized to a rule? Can you predict the next one? What about the last one? Does it work for other starting values?

Fig 8.1. Common Core mathematical practices (NGA Center and CCSSO 2010, p. 6) and questioning prompts

The following research lesson was part of the professional development program. Using multiple data sources, we wanted to capture teachers' strategic competence, namely the connection between teachers' mathematical knowledge and practices and their use of students' representations and diverse strategies to move the mathematics agenda forward. The researchers included mathematics educators and mathematics specialists who recorded their observations as research memos from the follow-up meetings and lesson study. These observations, along with teachers' reflections, were then used to analyze and identify recurring themes from lessons. In addressing our research questions, we found that the two questions were closely linked, in that, as we examined how teachers used representations and models to elicit strategic competence in students, we identified specific kinds of mathematical knowledge and instructional practices that came into play as teachers focused on developing students' strategic competence. Using these notable teaching instances as examples of the use of representations to elicit strategic competence, our analysis showed four key themes that illustrated the ways in which teachers used representations and models and how their mathematics knowledge for teaching developed as they promoted strategic competence and the mathematical practices. We will describe these themes with classroom examples and note how the mathematical practices played out in the lessons.

Formulating a Rich Task

One major theme that we found was that teachers spent a majority of their planning time formulating a rich problem that would elicit diverse strategies and conceptual understanding of proportional reasoning. For example, one of the research lessons we designed with a group of fourth-through-eighth-grade teachers and two mathematics specialists was called the Mango problem. The task provided opportunities for students to engage in meaningful mathematics and make sense of the problem and persevere in solving it (MP.1). Here is the problem:

> One night, the king couldn't sleep, so he went down into the royal kitchen, where he found a bowl full of mangoes. Being hungry, he took $1/6$ of the mangoes from the bowl. Later that same night, the queen was hungry and couldn't sleep. She, too, found the mangoes and took $1/5$ of what the king had left. Still later, the prince awoke, went to the kitchen, and ate $1/4$ of the remaining mangoes. Even later, his sister, the princess, ate $1/3$ of what was then left. Finally, the royal dog woke up hungry and ate $1/2$ of what was left, leaving only 3 mangoes for the kitchen staff. How many mangoes were originally in the bowl?

Teachers attended to precision (MP.6) as they debated on the wording of the fraction problem (i.e., $1/6$ of the bowl of mangoes, $1/5$ of what the king had left.) and how they would introduce it. They wanted to introduce the problem so that students would understand the task, yet the cognitive demand was not stripped away. For example, teachers decided in the planning meeting that students should "act out the problem" or model the mathematics (MP.4) so that the teachers could be sure the students understood and could visualize the problem. Students were given independent time so that all students could "own" the problem by making sense of it first. In addition, the teachers wanted to assess how individuals were making sense of quantities and their relationships in the problem situation (MP.2) before the group conversation.

Some of the students held a mental model of fractional parts of the mango as a region model and ran into problems when they arrived at the last part of the problem where "the royal dog ate $1/2$ of what was left, leaving only 3 mangoes." This problem required students to have a more sophisticated understanding of fractions, including the set model and unitizing fractions. The problem also required students to attend to precision (MP.6), as they needed to communicate what one-sixth of the bowl meant when referring to part of a set.

A seventh-grade teacher who retaught this lesson after the first cycle, which was hosted by the fourth-grade teacher, reflected on how the older students had a better understanding of the multiple meanings of fractions and were more flexible in unitizing a set as part of a whole. That is, they demonstrated their understanding of the set model for fractions and understood what it meant when they interpreted "$1/2$ of the bowl of mangoes equals 3 mangoes" as the question, "What is $1/2$ of $x = 3$?" and could do and undo what was happening in the story problem. They used repeated reasoning (MP.8) and saw an algebraic formula. For example, they understood that if 6 mangos is $2/3$ of the mangoes left before the princess took $1/3$, then to find the whole, they must find $2/3$ of $x = 6$, solve for x, and get 9. They were able to continue: if $3/4$ of $x = 9$, then the whole is 12. Essentially, they were *doing and undoing an operation,* one of the algebraic habits of mind (Driscoll 1999). This illustrated that the task had an important mathematical agenda embedded, which was understanding unitizing fractions, the set model for fractions, and the algebraic habits of mind of doing and undoing. The task allowed teachers to analyze which students in their class had a more robust and flexible understanding of fractions as they observed students unitize and chunk a fraction of the set as a fractional part. The challenging aspect of this problem was that the unit, in this case one whole bowl of mangoes, was the unknown that students needed to find.

Using Students' Diverse Strategies as Pedagogical Content Tools

A second major theme we found is that the teachers became more intentional about monitoring students' thinking (Stein and Smith 2011) so that they could use students' strategies or representations as pedagogical tools for classroom discourse. In the professional development, we focused on how teachers could use tools like student artifacts and representations (e.g., diagrams, manipulative models, small-group discussions, numeric notations) to discuss important mathematical ideas. In addition, we emphasized Stein and Smith's five practices for orchestrating mathematics discourse (2011), which are anticipating what students will do, monitoring their work in class, selecting student work to use in discussion, sequencing those students' work, and connecting the strategies to big ideas in mathematics.

While planning and anticipating student strategies, teachers noted in their lesson plans that they would look for students to use a general method or express repeated reasoning (MP.8), such as students making six equal groups with three mangoes in each group as shown in figure 8.2.

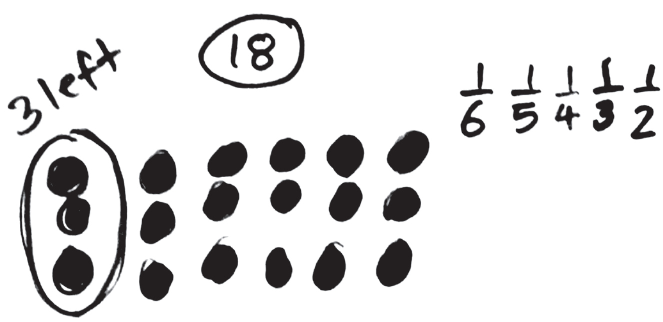

Fig. 8.2. Student strategy: Unitizing

In addition, while observing the lesson, teachers noticed that students were making use of structures (MP.7) as they related multiplication and division of fractional parts as they thought about multiples of threes or sixes, since three was the number of mangos remaining in each group and six was the number of groups. Teachers noted that students would need to realize that the amount that was taken each time was the same portion. The important idea was to see if students were able to unitize by constructing a reference unit.

In the following excerpt, the lead teacher allows students to engage in meaningful mathematical conversation as they share different strategies (MP.2, MP.3). To show how the teacher used the mathematical practices to facilitate the mathematics conversation, we have identified some of the relevant practices:

S1: I doubled the leftover 3 × 2 since 3 + 3 = 6. Then I did $1/3$ of 6, which is 2 and then added that to 6 to get 8. Then I did $1/4$ of 8, which is 2 and added to 8 to get 8 + 2 = 10. Then I took $1/5$ of 10 which is 2 and added to get 10 + 2 = 12 and did $1/6$ of 12 and added it to 12 to get 12 + 2 to get 14.

T: Could you show it with the manipulatives and act it out? [*Eliciting modeling mathematics (MP.4)*]. How did others approach this problem? What are we doing here?

S2: Now I am figuring out another way. First I did 3 times 2 because he did $1/2$. So you get 6. Then we have to do 6 times 3 for $1/3$. Then you get 18. We do 18 times 4 for $1/4$, which is 72. Then you have to do 72 times 5, which is . . .

[*Teacher observer's note: At this time, S2 gets full approval from S1, and they keep multiplying to get 2,160 mangoes. Of course S3 is not convinced.*]

S3: How many did the king eat?

S1 and S2: He ate $1/6$ of that. He ate 360.

T: Do you think that is a reasonable amount? [*Here the teacher prompts students to construct viable arguments and critique the reasoning of others (MP.3)*]

[*Teacher observer's note: Then student S3 decides that it is time to convince everyone at the table with his approach. He puts down three snap cubes.*]

S3: The royal dog ate 3. [*S3 puts down three more cubes making the total cubes 6 on the table.*] So this is how it was before royal dog came along. Before the royal dog, there were 6 mangoes. [*Student attempts to make sense of the problem and continues to explain (MP.1).*]

T: I have a question for the group. Did the royal dog eat less, more, or the same as the others? How do you know?

S3: Before the royal dog ate, this was how many were left [*grabbing all 6 cubes*] and this is how many he ate [*dropping 3 cubes down*] with 3 left.

T: OK. [*seems convinced*] Go on—

S3: [*Reads the problem*] Princess, it says, ate $1/3$ of what was then left. There were nine mangoes before she ate and since there are 6 after, which means she ate 3. [*At this point he drops 3 more cubes down*].

T: How do you know that there were 9 before she ate?

S3: [*Shows three sets of three with the cubes; see fig. 8.3.*] I am working back in time. They took the same amount each time because the king took $1/6$ of the whole and that left 5 groups, the queen took $1/5$ of the whole bowl and that left 4 groups, and the prince took $1/4$ of the bowl and left 3 groups and the princess took $1/3$ of the whole which left 2 groups and the royal dog took $1/2$ of it leaving 3. [*The student attends to precision while explaining to classmates (MP.6)*].

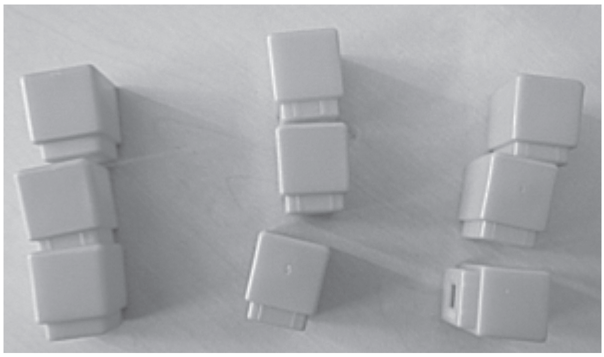

Fig. 8.3. Using manipulatives to act out the problem

[*S3 is busy convincing the group and the teacher. He grabs all nine cubes on the table.*]

S3: This is before the prince ate. And then the prince came, he ate $1/4$ of what was before him which means 9 is left after he ate [*points the 9 cubes on the table*]. So that means there was 12. Before the queen came, there were 15, and then she ate it, and there were 12. So before the king ate his portion of $1/6$, there must have been 18. [*Student takes the time to reason quantitatively (MP.2)*].

[*At this point S3 seems to have convinced* everyone!]

S2: Actually, I think [S3] is right!

This episode illustrated an important aspect of eliciting strategic thinking among students through questioning and giving students time and space to make arguments and convince one another. The important move for the teacher was to be a keen listener and to allow students space for social construction of knowledge. This excerpt captures how student thinking can easily be misled by other students and how tools can be helpful if used appropriately. It also demonstrates how misconceptions can prevail until someone (like a teacher or a fellow student) or something (like a probing question or multiple problem-solving approaches such as pictures, demonstrations with manipulatives, and verbal explanations) helps not only clarify misconceptions but also solve the problem efficiently.

Recognizing That Different Tools Lead to Diverse Thinking

The third theme is that teachers noticed tools could help or hinder interpretation during the problem-solving process. A tool was only helpful if the learner could attach meaning or match their mental interpretation with the manipulatives. From the first cycle, teachers noticed how offering students access to multiple manipulatives actually hindered their thinking. In the previous excerpt, a student had success with connecting cubes and used them to model the portion taken by each of the characters. However, there were some students who chose the fraction circles and had a difficult time because they could not make their mental picture of three remaining mangos fit into the regional model of fraction circles. They wanted the fraction circles to represent the three remaining mangoes in the bowl and the fractional parts that the characters took out of the bowl, but the students did not see how three discrete mangos could be represented by the continuous regions of the fraction circle. After observing the first research lesson, one of the observers, another fourth-grade teacher, decided to experiment by limiting the available manipulatives to see how students would represent their thinking with the selected tools. But then she wondered whether it would limit some students' development of decision making because she felt that choosing appropriate tools (MP.5) was an important part of strategic competence.

> First, I reduced the number of manipulatives available in the toolkits. I restricted my toolkits to multilink cubes, two-color counters, and die-cuts of mango fruits. The positive [aspect of this approach was that] students were not as likely to continue moving along from manipulative to manipulative in an effort to have the manipulative solve the problem for them. The negative [aspect of this approach was that] it is possible that the manipulative with which the student had the most familiarity and confidence in was not represented in the toolkit. (A fourth-grade teacher)

Another teacher who taught the lesson in her own classroom decided to allow students to draw their own pictures (see fig. 8.4) to represent their mental images of the problem and found that these students were more successful. In these diagrams, teachers noticed how students showed their understanding of unitizing.

Teachers' ability to experiment through multiple research lessons allowed us to conclude that good problem solvers who had a strong grasp of the fraction concept had flexible mental models to represent the problem, whether it was through making their own drawings or using manipulatives. That is, while some struggled to model the problem with the choice of fraction circles, others who had an understanding of unitizing fractions were able to represent each fractional piece as three mangos and chunk them to represent a fractional part—modeling their thinking. The success of the use of tools was not so much the choice of tools, such as tiles or fraction circles, rather the types of mental models students had of fractions that allowed them to be effective in translating the problem situation with manipulatives and other tools for thinking (MP.1 and MP.4).

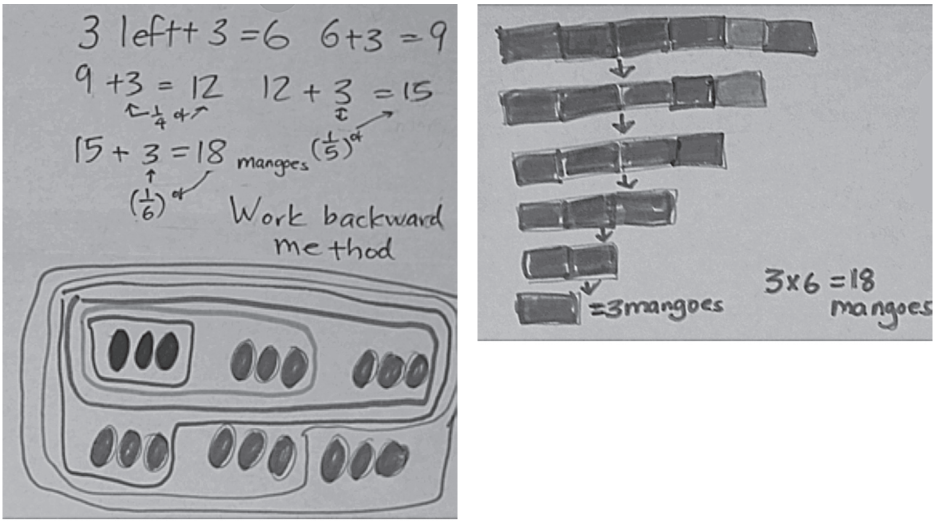

Fig. 8.4. Students' multiple approaches

■ Using Questioning and Mathematics Discourse

With the focus on analyzing students' strategic competence, teachers anticipated students' solution strategies (Stein and Smith 2011; Lewis 2002) To elicit students' strategic competence, the group of teachers spent time generating probing questions to pose during independent work and group math talk that encouraged students to defend their reasoning (MP.3) and to attend to precision (MP.6). This is seen in one fourth grade teacher's reflection:

> As I monitored their work, I asked open-ended questions like "Why have you settled on the original amount to be 18?" Or, "Did they have whole mangoes or parts of mangoes?" when students were not clear on what a unit should be. And often, the question was simply, "What does this drawing or statement mean?" My questions helped students re-gather information that they may have overlooked in the problem. Often, I asked probing questions that helped students explain and understand their thinking. My monitoring showed what groups of students were thinking alike, concretely, abstractly, and illustratively.

This teacher noted that students who made sense of the problem could model the problem (MP.4) and explain their thinking confidently, their models matching or proving their thinking. However, students who were incorrect in their solutions were typically dissatisfied with their models, as if something were missing that they were unable to explain. And these latter students felt their results were unreasonable or the results lacked adequate proof, so they were confident that they were wrong. However, these students were unable to explain why their solution was wrong. To extend students' thinking, this teacher asked students to share their responses within the group and viewed each group's work. Then each group explained their approach to the problem (MP.3). The teacher sequenced their work for display and discussion (Stein and Smith 2011) starting with concrete models, followed by the guess-and-check model and the logical work-backward model that exhibited algebraic thinking. The groups of students that unitized the mangoes displayed concrete models. The guess-and-check approach revealed some reasonable estimates and reasoning skills because many of them noted that they looked for a pattern and tried different multiples of three until they found the one that worked. The "work backward" approach was a logical model where students used their algebraic thinking to undo the operation by adding three each time. The class discussed the connections and differences between each of the three strategies as well as their efficiency and effectiveness. The success of the lesson was to move students to unitize the mangoes, which required students to reformulate quantities in smaller chunks. Students approached this idea in multiple ways: the work-backward strategy, the concrete strategy of using manipulatives and drawings to show the partitioning of the whole, and the guess-and-check method. The effective use of questioning and mathematics discourse in the classroom allowed students' strategic competence to take center stage and moved more students toward an understanding of efficient and advanced strategies for the problem.

Developing Students' and Teachers' Strategic Competence through the Mathematical Practices

We found through our research lesson that a focus on mathematical practices facilitated the development of teachers' and students' strategic competence. We now revisit the eight mathematical practices and how they helped teachers and students develop strategic competence. First, teachers selected a rich task for students to make sense of problems and persevere in solving them (MP.1) and sustained their interest in the problem-solving process by engaging them in mathematics conversation where they discussed the reasoning of their classmates (MP.3). Teachers focused on having students reason abstractly and quantitatively (MP.2) through the Mango problem—exploring the set model of fractions and the concept of unitizing fractions. Students modeled mathematics by visualizing the problem situation (MP.4) and used appropriate tools to represent their thinking (MP.5). Students used mathematical structures (MP.7), such as division and multiplication of fractional parts of a set, to look for patterns and methods in the repeated reasoning (MP.8). Some of the students found algebraic strategies of doing and undoing what the characters took in the story problem. Teachers offered the time and space for students to critique classmates' reasoning and value different perspectives to problem solving. In addition, teachers developed their strategic competence as they evaluated appropriate uses of tools and used them judiciously as pedagogical content tools to help connect students' mathematical thinking.

Developing proficiency in teaching mathematics with a focus on eliciting students' strategic competence required more than the analysis of students' diverse strategies. It also required providing time and space for students to reason by sharing, arguing, and justifying their strategic thinking. Through the experience of working together, teachers helped each other gain an appreciation for multiple strategies. As the teachers in the study relearned mathematics through multiple models, they felt more confident and more strategically competent using multiple representations and strategies while posing rich proportional reasoning problems and engaging in meaningful mathematics discourse in class with their students.

Acknowledgments

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page 88 blank